MIT OCW GR PSET 6

1 · Constraint + evolution equations

· In E + M, Maxwell's equations are:

$\partial_i E^i = 4\pi \rho, \; \partial_i B^i = 0;$ (i)

$\partial_t E^i = \varepsilon^{ij}{}_k \partial_j B^k - 4\pi J^i, \; \partial_t B^i = -\varepsilon^{ij}{}_k \partial_j E^k$ (ii)

· The first set of equations (i) are static in time + represent "constraints". The 2nd set of equations (ii) show how E^i and B^i "evolve" in time.

· We want to show that a similar decomposition exists for the EFE $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ ....

· Show that $G_{0\nu} = 8\pi G T_{0\nu}$ has the LHS $G_{0\nu}$ containing at most 1 ∂_t time derivative + these set of 4 equations represent the constraints whereas the other 6 equations contain up to order ∂_t^2 + therefore represent the evolution of time:

• Note that the following proof is essentially stolen from Carroll's lecture notes. Check them out on the Arxiv...

• Begin with the Bianchi Identity / Fact that divergence of Einstein tensor is zero:

$\nabla_\sigma G^{\nu\sigma} = 0$ Now expand this out...

$\nabla_0 G^{0\sigma} + \nabla_i G^{i\sigma} = 0$, i.e. time + spatial

$$\rightarrow \partial_0 G^{0\sigma} + \partial_i G^{i\sigma} + \Gamma^{\nu}_{\nu\lambda} G^{\lambda\sigma} + \Gamma^{\sigma}_{\nu\lambda} G^{\nu\lambda} = 0$$

sum over 1 G component & 1 Γ component

sum over the 2 G components

$$\rightarrow \partial_t G^{0\sigma} = -\partial_i G^{i\sigma} - \Gamma^{\nu}_{\nu\lambda} G^{\lambda\sigma} - \Gamma^{\sigma}_{\nu\lambda} G^{\nu\lambda}$$

assuming a diagonal metric $[g_{\nu\sigma}]$ and applying the Carrol Γ identities, you'll find that the Gamma terms are at most of the order $\partial_t g_{\mu\nu}$ and we assume $G^{\lambda\nu}$ is at most of the order $\partial_t^2 g_{\mu\nu}$ ... Therefore, the RHS contains terms like $(\partial_t g_{\mu\nu})(\partial_t^2 g_{\mu\nu})$ or $(\partial_t^2 g_{\mu\nu})$ or $(\partial_t g_{\mu\nu})$ but no higher (≥ 3) time derivatives like $\partial_t^3 g_{\mu\nu}$ ...

This implies $\partial_t G^{0\nu} \propto O(\partial_t^2 g_{\mu\nu})$ which in turn implies $G^{0\nu} \propto O(\partial_t g_{\mu\nu})$

So we have shown that $G^{0\nu}$ contains at most 1 time derivative of $[g_{\mu\nu}]$ and these set of 4 equations represent the constraint equations for the system of Einstein's Field Equations ✓

Q.E.D. ✓

$\boxed{2}$ Action for a cosmological constant

• Show that varying the action:

$$S = \int d^4x \sqrt{-g}\,(R + a)$$

where R is the Ricci scalar and "a" is a constant yields the Einstein equation with a cosmological constant. How does "a" relate to the cosmological constant "Λ"?

Givens:

$$R \equiv g^{\alpha\beta} R_{\alpha\beta}$$

Require that $\delta S = 0$:

$$\rightarrow \quad 0 = \delta S = \int d^4x \frac{\delta}{\delta g^{\alpha\beta}} \left[\sqrt{-g}\,(g^{\alpha\beta} R_{\alpha\beta} + a) \right] \delta g^{\alpha\beta}$$

• By Leibniz Rule:

$$\delta\left(\sqrt{-g}\,(g^{\alpha\beta} R_{\alpha\beta} + a)\right) = \left(\delta\sqrt{-g}\right)\left(g^{\alpha\beta} R_{\alpha\beta} + a\right) + \sqrt{-g}\, R_{\alpha\beta}\,\delta g^{\alpha\beta} + g^{\alpha\beta}\sqrt{-g}\,\delta R_{\alpha\beta}$$

• From lecture we were given that:

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{\alpha\beta}\,\delta g^{\alpha\beta}$$

· And $g^{\alpha\beta}\,\delta R_{\alpha\beta} \equiv \nabla_\alpha V^\alpha$

$$\rightarrow 0 = \delta S = \int d^4x \Big[-\tfrac{1}{2}\sqrt{-g}\, g_{\alpha\beta}\left(g^{\alpha\beta}R_{\alpha\beta} + a\right)\delta g^{\alpha\beta} + \sqrt{-g}\,R_{\alpha\beta}\,\delta g^{\alpha\beta} + \sqrt{-g}\,\nabla_\alpha V^\alpha \Big]$$

$\emptyset$ by div. Thrm.
@ infinity...

$$\rightarrow 0 = \int d^4x \sqrt{-g}\Big[\underbrace{-\tfrac{1}{2} g_{\alpha\beta} R + R_{\alpha\beta}}_{\equiv G_{\alpha\beta}} - \tfrac{a}{2} g_{\alpha\beta}\Big]\delta g^{\alpha\beta}$$

$$\rightarrow G_{\alpha\beta} - \frac{a}{2} g_{\alpha\beta} = 0$$

$$\rightarrow \boxed{G_{\alpha\beta} + \Lambda g_{\alpha\beta} = 0 \quad \text{where} \quad \Lambda \equiv -\frac{a}{2}}$$

$\boxed{3}$ Nordstrøm's Gravity Theory

- A metric theory devised in 1913 relates $g_{\mu\nu}$ + $T_{\mu\nu}$ by:

$$C_{\alpha\beta\gamma\delta} = 0 \quad ; \quad R = K g_{\mu\nu} T_{\mu\nu} \equiv KT$$

↑ "Weyl Tensor"

- This system is conformally flat meaning the metric is given by:

$$g_{\mu\nu} = e^{2\phi}\eta_{\mu\nu}$$

where $\phi = \phi(x^{\mu})$ is a function of the spacetime coordinates.

$\boxed{a}$. show that for $\phi^2 \ll 1$ and $|\partial_t \phi| \ll |\partial_i \phi|$ the geodesic equation for a slowly moving test body ($v^i \ll 1$) in this spacetime reproduces the kinematics of Newtonian Gravity:

$$\rightarrow \quad \vec{u} \approx (1, \vec{0})$$

$$\frac{du^{\alpha}}{d\tau} + \Gamma^{\alpha}_{tt} u^t u^t = 0$$

$$\rightarrow \quad \frac{du^{\alpha}}{d\tau} = -\Gamma^{\alpha}_{tt}$$

$$\rightarrow \frac{d^2x^\alpha}{d\tau^2} = -\Gamma^\alpha_{tt} = \frac{1}{2}(g_{\alpha\alpha})^{-1}\partial_\alpha g_{tt}$$

· So for spatial coords:

$$\frac{d^2x^i}{d\tau^2} = \frac{1}{2}e^{-2\phi}\partial_i(-e^{2\phi})$$

$$= -\frac{1}{\cancel{2}}\cancel{e^{-2\phi}}(\cancel{2}\partial_i\phi)\cancel{e^{2\phi}} = -\partial_i\phi$$

$$\rightarrow \boxed{\frac{d^2x^i}{d\tau^2} = -\partial_i\phi \quad \text{i.e.} \quad a \propto -\vec{\nabla}V(x) \quad \text{Newtonian Limit}}$$

[b]. Show that in this Newtonian limit, the Ricci Scalar is just a 2nd order Diff. Eq. acting on ϕ:

$$R \equiv g^{\mu\nu}R_{\mu\nu}$$

$$= g^{\mu\nu}(\partial_\lambda\Gamma^\lambda_{\mu\nu} - \partial_\nu\Gamma^\lambda_{\lambda\mu} + \Gamma^\sigma_{\mu\nu}\Gamma^\lambda_{\lambda\sigma} - \Gamma^\sigma_{\mu\nu}\Gamma^\lambda_{\nu\sigma})$$

· The $\Gamma\Gamma$ terms (if you compute them ☺) are proportional to $(\partial_i\phi)^2$ but let these go to zero in this limit since we care about $\partial_i^2\phi$ type terms to recover a 2nd order differential operator. ⟶

$\rightarrow R \propto g^{\nu\sigma}\left(\partial_\lambda \Gamma^{\lambda}_{\nu\sigma} - \partial_\sigma \Gamma^{\lambda}_{\lambda\nu}\right)$

$g^{\nu\sigma} = e^{-2\phi}\eta^{\nu\sigma}$ which is diagonal. Therefore, by the Carroll identities:

$$\Gamma^{\lambda}_{\nu\nu} = -\frac{1}{2}(g_{\lambda\lambda})^{-1}\partial_\lambda g_{\nu\nu}$$

$$\Gamma^{\lambda}_{\lambda\nu} = \partial_\nu \ln\left(\sqrt{|g_{\lambda\lambda}|}\right)$$

· only $g^{tt}, g^{xx}, g^{yy}, g^{zz} \neq 0$

$$g^{tt} = e^{-2\phi}\eta^{tt} = -e^{-2\phi}\eta^{ii}$$

· Break it down term by term:

$$g^{\nu\sigma}\partial_\lambda\Gamma^{\lambda}_{\nu\sigma} = g^{tt}\partial_\lambda\Gamma^{\lambda}_{tt} + g^{ii}\partial_\lambda\Gamma^{\lambda}_{ii}$$

$$= \underbrace{g^{tt}\cancel{\partial_t}\Gamma^{t}_{tt}}_{0} + g^{tt}\partial_i\Gamma^{i}_{tt} + \underbrace{g^{ii}\cancel{\partial_t}\Gamma^{t}_{ii}}_{0} + g^{ii}\partial_j\Gamma^{j}_{ii}$$

since $|\partial_t\phi| << |\partial_i\phi|$ ...

$$\propto g^{tt}\partial_i\Gamma^{i}_{tt} + g^{ii}\partial_j\Gamma^{j}_{ii}$$

$$\Gamma^{i}_{tt} = -\frac{1}{2}(g_{ii})^{-1}\partial_i g_{tt} \longrightarrow$$

$$\Gamma^i_{tt} = \left(-\tfrac{1}{2}\right)\left(e^{-2\phi}\right)\partial_i\left(-e^{2\phi}\right) = \partial_i \phi$$

$$\Gamma^j_{ii} = \left(-\tfrac{1}{2}\right)\left(e^{-2\phi}\right)\partial_j\left(e^{2\phi}\right) = -\partial_j \phi$$

$$\rightarrow g^{\mu\nu}\partial_\lambda \Gamma^\lambda_{\mu\nu} \approx \left(e^{-2\phi}\right)\left(\partial_i^2 \phi - 3\partial_i^2 \phi\right)\eta^{ii}$$

flip sign as well to flip $tt \rightarrow ii$

3 since sum over spatial components...

$$\rightarrow g^{\mu\nu}\partial_\lambda \Gamma^\lambda_{\mu\nu} \approx -2e^{-2\phi}\Box\phi$$

· Now $-g^{\mu\nu}\partial_\nu \Gamma^\lambda_{\lambda\mu}$

$$= -g^{tt}\cancelto{0}{\partial_t}\Gamma^\lambda_{\lambda t} - g^{ii}\partial_i \Gamma^\lambda_{\lambda i}$$

$$= -g^{ii}\partial_i \Gamma^t_{ti} - g^{ii}\partial_i \Gamma^j_{ji}$$

$$\Gamma^t_{ti} = \partial_i \ln\left(\sqrt{|g_{tt}|}\right) = \partial_i\left(\ln\left(\sqrt{e^{2\phi}}\right)\right) = \partial_i \phi$$

$$\Gamma^j_{ji} = \Gamma^t_{ti} = \partial_i \phi$$

$$\rightarrow -g^{\mu\nu}\partial_\nu \Gamma^\lambda_{\lambda\mu} = \left(-e^{-2\phi}\right)\left(\partial_i^2\phi + 3\partial_i^2\phi\right)\eta^{ii} = -4e^{-2\phi}\Box\phi$$

$\rightarrow \quad R = g^{\nu\sigma}\left(\partial_\lambda \Gamma^\lambda_{\nu\sigma} - \partial_\sigma \Gamma^\lambda_{\lambda\nu}\right)$

$= \left(-e^{-2\phi}\right)\left(2\Box\phi + 4\Box\phi\right)$

$\rightarrow \boxed{\begin{array}{l} R = -6e^{-2\phi}\,\Box\phi = D^2\phi \\ \text{where} \quad D^2 = -6e^{-2\phi}\,\Box \end{array}}$

$\boxed{c}$. Show that Nordstrom's field equation reduces to Newtonian gravitation in the proper limits:

. Nordstrom's equations:

$$R = K g^{\nu\sigma} T_{\nu\sigma} = -6e^{-2\phi}\,\Box\phi$$

$$e^{-2\phi} \approx 1 - \cancel{2\phi}_{\to 0} \approx 1$$

$$\rightarrow \Box\phi \approx -\frac{K}{6} g^{\nu\sigma} T_{\nu\sigma} \leftarrow T_{\nu\sigma} \approx \begin{pmatrix} \rho & & & 0 \\ & 0 & & \\ & & 0 & \\ 0 & & & 0 \end{pmatrix}$$

$$\rightarrow \Box\phi = -\frac{K}{6} g^{tt} T_{tt} = -\frac{K}{6}\left(-e^{-2\phi}\right)\rho \approx \frac{K\rho}{6}$$

. compare this to $\nabla^2\phi = 4\pi G\rho$ for Newton $\rightarrow$

$$\Box \equiv \eta^{\nu\sigma}\partial_\nu\partial_\sigma \approx \eta^{ii}\partial_i\partial_i$$

$$= \eta^{ii}\partial_i^2 \quad \text{since} \quad |\partial_i\phi| \gg |\partial_t\phi|$$

$$= e^{-2\phi}\partial_i^2 \approx \partial_i^2$$

$$\boxed{\begin{array}{l} \rightarrow \partial_i^2\phi \approx 4\pi G\rho \quad \text{as long as} \quad K = 24\pi G \\ \rightarrow R = 24\pi G T \end{array}}$$

$\boxed{d}$. Is this theory consistent with the Pound-Rebka gravitational redshift experiment?

- My attempt at a logical confirmation:
- We are given $g_{\nu\sigma} = e^{2\phi(x^\nu)}\eta_{\nu\sigma}$

and know that $g_{\nu\sigma}u^\nu u^\sigma = \vec{u}\cdot\vec{u} = -1$ for a massive observer with 4-velocity $\vec{u}$

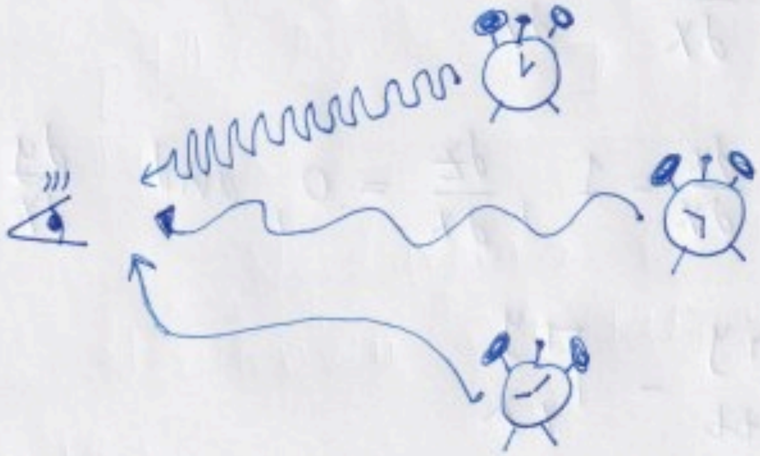
- Let's assume stationary observers: $\vec{u} = (u^t, \vec{0})$ with no assumption on spatial position (x, y, z)

$$\rightarrow g_{tt}u^t u^t = -1$$

$$\rightarrow -e^{2\phi}\left(\frac{dt}{d\tau}\right)^2 = -1 \rightarrow \frac{dt}{d\tau} = e^{-\phi}$$

$\rightarrow \; d\tau = e^{\phi} dt$

and since $\phi = \phi(x^\nu)$ is a function of the coords, implies clocks at different positions in this spacetime tick at different rates for a CSO:



· Different tick rates imply different perceived frequencies for the same laser pointers...

· So qualitatively we can expect that the relativistic effects of this spacetime can indeed cause frequency shifts of light ✓✓

[e] · Show that there is no deflection of light by the sun in this theory of gravity:

· We will use the same prescription for solving the angle of deflection as in the last PSET:

$\rightsquigarrow$

• Start with the Geodesic equation:

$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\nu\sigma} \frac{dx^\nu}{d\lambda} \cdot \frac{dx^\sigma}{d\lambda} = 0$$

• Use $\alpha = y$ + choose $\lambda = x$:

$$\rightarrow \frac{d^2y}{dx^2} + \Gamma^y_{\nu\sigma} \frac{dx^\nu}{dx} \cdot \frac{dx^\sigma}{dx} = 0$$

$$\frac{dt}{dx} = \frac{1}{c} = 1 \quad , \frac{dx}{dx} = 1 \quad , \frac{dz}{dx} = 0, \text{ and } \frac{dy}{dx} \approx 0$$

$$\rightarrow \frac{d^2y}{dx^2} = -\Gamma^y_{tt} - \Gamma^y_{xx}$$

$$\Gamma^y_{tt} = \partial_y \phi \quad \text{and} \quad \Gamma^y_{xx} = -\partial_y \phi$$

$\rightarrow \frac{d^2y}{dx^2} = 0$ now integrate

$$\rightarrow \Delta\phi \equiv \frac{dy}{dx} = \int 0 \, dx = 0$$

$\rightarrow$ $\boxed{\Delta\phi = 0 \text{ which is inconsistent with experiment}}$ ₩

$\boxed{4}$ · An object of mass "m" is at rest on a bathroom scale in a weak gravitational field. The object has fixed (x, y, z) and the metric is given by $g_{\mu\nu} = \eta_{\mu\nu} + 2\phi \cdot \mathrm{diag}(1,1,1,1)$. We take $\phi^2 \ll 1$, $\partial_z \phi = -g$ and $\partial_\nu \phi = 0$ for $\nu \neq z$. Neglect $\phi^2 + g\phi$.

In this problem we will show that if one wants to interpret gravity as a force rather than as the effects of spacetime curvature, then it must be a velocity dependent force.

$\boxed{a}$. What force does the bathroom scale apply on the body?

· The EOM for the body is:

$$m \frac{D^2 x^\alpha}{d\tau^2} = m u^\beta \nabla_\beta u^\alpha = F^\alpha$$

$$\rightarrow F^\alpha = m u^\beta (\partial_\beta u^\alpha + \Gamma^\alpha_{\beta\gamma} u^\gamma)$$

$$u^{x,y,z} = 0$$

$$\rightarrow F^\alpha = m \Gamma^\alpha_{tt} (u^t)^2$$

$$\Gamma^\alpha_{tt} = -\frac{1}{2}(g_{\alpha\alpha})^{-1} \partial_\alpha g_{tt} \rightarrow \Gamma^x_{tt} = \Gamma^y_{tt} = \Gamma^t_{tt} = 0$$

$$\rightarrow F^z = \left(-\frac{m}{2}\right)(1+2\phi)^{-1}\left(\partial_z(2\phi-1)\right)(u^t)^2$$

$$F^z = \frac{mg\,(u^t)^2}{1+2\phi}$$

· Now calculate u^t:

$$-1 = g_{\alpha\beta}u^\alpha u^\beta = g_{tt}(u^t)^2 = (2\phi-1)(u^t)^2$$

$$\rightarrow (u^t)^2 = \frac{1}{1-2\phi}$$

$$\rightarrow F^z = \frac{mg}{(1+2\phi)(1-2\phi)} = \frac{mg}{1-4\phi^2} \approx mg$$

$$\rightarrow \boxed{\vec{F} \approx (0,0,0,mg)}$$ as one would expect...

[6] · Now suppose the object moves with constant 3-velocity $v = dx/dt = (dx/d\tau)(dt/d\tau)^{-1}$ in the x-direction:

$$V^x = vV^t \; ; \; V^y = V^z = 0$$

· What is V^t? While the mass is on the bathroom scale, what force does the scale apply to the mass?

$$\vec{V} = V^t(1, v, 0, 0)$$

$$\vec{V}\cdot\vec{V} = -1 = g_{\alpha\beta} V^\alpha V^\beta$$

$$\rightarrow\ -1 = g_{tt}(V^t)^2 + g_{xx} v^2 (V^t)^2$$

$$\rightarrow -1 = \left((2\phi - 1) + (1+2\phi)v^2\right)(V^t)^2$$

$$\rightarrow \boxed{V^t = \left((1-2\phi) - (1+2\phi)v^2\right)^{-1/2}}$$

$$F^\alpha = m u^\beta \nabla_\beta u^\alpha$$

$$= mV^x\left(\partial_x u^\alpha + \Gamma^\alpha_{x\nu} u^\nu\right) + mV^t\left(\partial_t u^\alpha + \Gamma^\alpha_{t\nu} u^\nu\right)$$

$$= (mV^t)\left(v\,\partial_x u^\alpha + v\,\Gamma^\alpha_{x\nu} u^\nu + \partial_t u^\alpha + \Gamma^\alpha_{t\nu} u^\nu\right)$$

· So now let's compute F^α component by component

$$F^t = (mV^t)\Big(\underbrace{\cancel{v\,\partial_x V^t}}_{0} + \underbrace{\cancel{\partial_t V^t}}_{0} + v\,\Gamma^t_{xx} V^x$$
$$+\, v\,\Gamma^t_{tx} V^t + \Gamma^t_{tx} V^x + \Gamma^t_{tt} V^t \Big)$$

$$\Gamma^t_{tx} = \Gamma^t_{xx} = \Gamma^t_{tt} = 0 \text{ since } \partial_x\phi = \partial_t\phi = 0$$

$\rightarrow \; F^t = 0$

$$F^x = (mv^t)\left(v\partial_x v^x + v\Gamma^x_{xx} v^x + v\Gamma^x_{xt} v^t + \partial_t v^x + \Gamma^x_{tt} v^t + \Gamma^x_{xt} v^x\right) = 0$$

and $F^y = 0$ similarly....

· So now we are left with only the z-component:

$$F^z = (mv^t)\left(\cancel{v\partial_x v^z}^{\,0} + v\Gamma^z_{xx} v^x + \cancel{v\Gamma^z_{tx} v^t}^{\,0} + \underbrace{\cancel{\partial_t v^z}}_{0} + \Gamma^z_{tt} v^t + \underbrace{\cancel{\Gamma^z_{tx}} v^x}_{0}\right)$$

$\downarrow$

$$F^z = (mv^t)\left(v^2 v^t \Gamma^z_{xx} + v^t \Gamma^z_{tt}\right)$$

$$\Gamma^z_{tt} = \Gamma^z_{xx} = \frac{g}{1+2\phi}$$

$$\rightarrow \; F^z = \frac{mg(v^t)^2(1+v^2)}{1+2\phi}$$

$$= \frac{(mg)(1+v^2)}{(1+2\phi)(1-2\phi-(1+2\phi)v^2)} = \frac{(mg)(1+v^2)}{1-(1+4\phi)v^2}$$

$$\rightarrow \boxed{\vec{F} = \left(0,0,0,\frac{mg(1+v^2)}{1-(1+4\phi)v^2}\right)}$$

· Which indeed is velocity dependent...

$\boxed{c}$ · Now transform coordinates by applying a naive Lorentz transformation along the x-axis. Evaluate the metric $g_{\bar{\nu}\bar{\nu}}$ in these new coords. To 1st order in ϕ, what are the force components in this new basis?

$$g_{\bar{\nu}\bar{\nu}} = \Lambda^{\alpha}_{\bar{\nu}} \Lambda^{\beta}_{\bar{\nu}} g_{\alpha\beta}$$

$$= \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^2 \begin{pmatrix} 2\phi-1 & 0 & 0 & 0 \\ 0 & 1+2\phi & 0 & 0 \\ 0 & 0 & 1+2\phi & 0 \\ 0 & 0 & 0 & 1+2\phi \end{pmatrix}$$

$$= \begin{pmatrix} \gamma^2+\gamma^2v^2 & -2v\gamma^2 & 0 & 0 \\ -2v\gamma^2 & \gamma^2+\gamma^2v^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2\phi-1 & 0 & 0 & 0 \\ 0 & 2\phi+1 & 0 & 0 \\ 0 & 0 & 2\phi+1 & 0 \\ 0 & 0 & 0 & 2\phi+1 \end{pmatrix}$$

$$\rightarrow \boxed{[g_{\bar{\nu}\bar{\nu}}] = \gamma^2 \begin{pmatrix} (2\phi-1)(1+v^2) & -2v(1+2\phi) & 0 & 0 \\ 2v(1-2\phi) & (2\phi+1)(1+v^2) & 0 & 0 \\ 0 & 0 & 2\phi+1 & 0 \\ 0 & 0 & 0 & 2\phi+1 \end{pmatrix}}$$

• Now how does F^{α} transform under $\Lambda^{\alpha}_{\ \bar{\beta}}$?

$$F^{\bar{\beta}} = \frac{\partial x^{\bar{\beta}}}{\partial x^{\beta}} F^{\beta} = \Lambda^{\bar{\beta}}_{\ \alpha} F^{\alpha}$$

$$= \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ F^{z} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ F^{z} \end{pmatrix}$$

$\rightarrow$ $F^{\bar{\beta}} = F^{\alpha}$ i.e. the Lorentz transform in the x-direction leaves the $\vec{F}$ components unchanged since only $F^{z} \neq 0$

$\boxed{d}$ · Show that the barred coordinate basis can be transformed to an orthonormal basis

$\vec{e}_{\hat{\nu}} = E^{\bar{\nu}}_{\hat{\nu}} \vec{e}_{\bar{\nu}}$ with a tetrad matrix:

$$E^{\bar{\nu}}_{\hat{\nu}} = \delta^{\bar{\nu}}_{\hat{\nu}} + \phi A^{\bar{\nu}}_{\hat{\nu}}$$

· Find $A^{\bar{\nu}}_{\hat{\nu}}$ and $F^{\hat{\nu}}$ to 1st order in ϕ:

· This problem took a while but I think one of the main things to beware of is that in GR an orthonormal basis satisfies

$\vec{e}_{\hat{\nu}} \cdot \vec{e}_{\hat{\gamma}} = \eta_{\hat{\nu}\hat{\gamma}}$ rather than $\delta_{\hat{\nu}\hat{\gamma}}$

· So let's begin with that postulate:

$$\eta_{\hat{\nu}\hat{\gamma}} = \vec{e}_{\hat{\nu}} \cdot \vec{e}_{\hat{\gamma}} = E^{\bar{\nu}}_{\hat{\nu}} E^{\bar{\gamma}}_{\hat{\gamma}} \vec{e}_{\bar{\nu}} \cdot \vec{e}_{\bar{\gamma}}$$

$$= E^{\bar{\nu}}_{\hat{\nu}} E^{\bar{\gamma}}_{\hat{\gamma}} g_{\bar{\nu}\bar{\gamma}}$$

· The matrix form of $[g_{\bar{\nu}\bar{\gamma}}]$ was found in the last problem. It technically is not diagonal, but I will assume V + γ are sufficiently small such that $g_{\bar\nu\bar\gamma} \propto \eta_{\bar\nu\bar\gamma} + 2\phi\,\delta_{\bar\nu\bar\gamma}$

- plugging this into our equality we get that:

$$\eta_{\hat\nu\hat\gamma} = \left(E^{\bar\nu}_{\hat\nu}\,E^{\bar\gamma}_{\hat\gamma}\right)\left(\eta_{\bar\nu\bar\gamma} + 2\phi\,\delta_{\bar\nu\bar\gamma}\right)$$

$$= \left(\delta^{\bar\nu}_{\hat\nu} + \phi A^{\bar\nu}_{\hat\nu}\right)\left(\delta^{\bar\gamma}_{\hat\gamma} + \phi A^{\bar\gamma}_{\hat\gamma}\right)\left(\eta_{\bar\nu\bar\gamma} + 2\phi\,\delta_{\bar\nu\bar\gamma}\right)$$

$$= \left[\delta^{\bar\nu}_{\hat\nu}\delta^{\bar\gamma}_{\hat\gamma} + \phi\left(A^{\bar\nu}_{\hat\nu}\delta^{\bar\gamma}_{\hat\gamma} + A^{\bar\gamma}_{\hat\gamma}\delta^{\bar\nu}_{\hat\nu}\right) + \underbrace{O(\phi^2)}_{\to\, 0}\right] \cdot \left[\eta_{\bar\nu\bar\gamma} + 2\phi\,\delta_{\bar\nu\bar\gamma}\right]$$

$$\propto \delta^{\bar\nu}_{\hat\nu}\delta^{\bar\gamma}_{\hat\gamma}\,\eta_{\bar\nu\bar\gamma} + \phi\left(A^{\bar\nu}_{\hat\nu}\delta^{\bar\gamma}_{\hat\gamma} + A^{\bar\gamma}_{\hat\gamma}\delta^{\bar\nu}_{\hat\nu}\right)\eta_{\bar\nu\bar\gamma} + 2\phi\,\delta^{\bar\nu}_{\hat\nu}\delta^{\bar\gamma}_{\hat\gamma}\,\delta_{\bar\nu\bar\gamma}$$

$$= \eta_{\hat\nu\hat\gamma} + 2\phi\,\delta_{\hat\nu\hat\gamma} + \phi\left(A^{\bar\nu}_{\hat\nu}\,\eta_{\bar\nu\hat\gamma} + A^{\bar\gamma}_{\hat\gamma}\,\eta_{\hat\nu\bar\gamma}\right)$$

$$\longrightarrow -2\delta_{\hat\nu\hat\gamma} = A^{\bar\nu}_{\hat\nu}\,\eta_{\bar\nu\hat\gamma} + \underbrace{A^{\bar\gamma}_{\hat\gamma}\,\eta_{\hat\nu\bar\gamma}}_{\underbrace{A^{\bar\nu}_{\hat\nu}\,\delta^{\bar\gamma}_{\bar\nu}\,\delta^{\hat\nu}_{\hat\gamma}\,\eta_{\hat\nu\bar\gamma}}_{A^{\bar\nu}_{\hat\nu}\,\eta_{\bar\nu\hat\gamma}}}$$

$$\rightarrow \quad -2\delta_{\hat{\nu}\hat{v}} = 2A^{\bar{\nu}}_{\hat{\nu}}\eta_{\bar{\nu}\hat{v}}$$

$$\rightarrow \quad -\delta_{\hat{\nu}\hat{v}} = A^{\bar{\nu}}_{\hat{\nu}}\delta^{\hat{\nu}}_{\bar{\nu}}\eta_{\hat{\nu}\hat{v}}$$

$$\rightarrow \quad A^{\bar{\nu}}_{\hat{\nu}} = -\delta_{\hat{\nu}\hat{v}}\eta^{\hat{\nu}\hat{v}}\delta^{\bar{\nu}}_{\hat{\nu}} - -\eta^{\bar{\nu}}_{\hat{\nu}}$$

$$\rightarrow \quad \boxed{A^{\bar{\nu}}_{\hat{\nu}} \approx -\eta^{\bar{\nu}}_{\hat{\nu}}}$$

. Now find $F^{\hat{\nu}}$:

$$F^{\hat{\nu}} = E^{\hat{\nu}}_{\bar{\nu}}F^{\bar{\nu}} =$$

$$= \left(\delta^{\hat{\nu}}_{\bar{\nu}} + \phi A^{\hat{\nu}}_{\bar{\nu}}\right)F^{\bar{\nu}}$$

$$= \left(\delta^{\hat{\nu}}_{\bar{\nu}} - \phi\eta^{\hat{\nu}}_{\bar{\nu}}\right)F^{\bar{\nu}}$$ now in matrix form

$$= \mathrm{diag}\{1+\phi, 1-\phi, 1-\phi, 1-\phi\}\begin{pmatrix} 0 \\ 0 \\ 0 \\ F^{\bar{z}} \end{pmatrix}$$

$\rightarrow \vec{F}^{\wedge} = (0, 0, 0, (1-\phi) F^{\bar{z}})$

- So compared to the barred coordinate basis, the $t, x, + y$ components are unchanged, but $F^{\hat{z}}$ is $(1-\phi) F^{\bar{z}}$ W

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5 "Geometrized units"

"Mass of sun" $\equiv M_{\odot} = 1.99 \times 10^{33}$ gm

"Newton constant" $\equiv G = 6.67 \times 10^{-8}\ cm^3 \cdot gm^{-1} \cdot sec^{-2}$

"Speed of light" $\equiv c = 3.00 \times 10^{10}$ cm / sec

· Do the following conversions:

a · Mass of the Earth in cm:

$M_{\oplus} \equiv$ "Mass of Earth" $= 5.98 \times 10^{27}$ gm

$$\boxed{M_{\oplus}^{Geom} \text{ in cm} = M_{\oplus} G / c^2 \approx 4.43 \times 10^{-1} \text{ cm}}$$

b · Density of neutron stars in cm^{-2}:

$\bar{\rho} = 10^{15}$ gm/cm^3

$$\rightarrow \boxed{\bar{\rho}^{\,Geom} \text{ in } cm^{-2} = \frac{\bar{\rho} G}{c^2} \approx 0.74 \times 10^{-13}\ cm^{-2}}$$

c · Pressure at core of a neutron star in cm^{-2}:

$P = 10^{34}\ gm \cdot sec^{-2} \cdot cm^{-1}$

$$\rightarrow \boxed{P^{Geom} = \frac{P G}{c^4} \approx 8.23 \times 10^{-16}\ cm^{-2}}$$

d . Acceleration due to gravity at the surface of Earth $g = 9.8\, m/s^2$ in sec^{-1} and $years^{-1}$

$$\boxed{g^{Geom} = \frac{100\, g}{c} \approx 3.27 \times 10^{-8}\ sec^{-1} \approx 0.1\ year^{-1}}$$

e . The typical luminosity of a gamma ray burst $L = 10^{53}\ erg/sec$ ($1\ erg = 1\ gm \cdot cm^2/sec^2$).

(In seconds I'm assuming?...)

$$\boxed{L^{Geom} = \frac{GL}{c^5} \approx 2.7 \times 10^{-7}\ sec}$$

f . The planck length^2 i.e. $\hbar$ in cm^2:

$$\boxed{\sqrt{\hbar^{Geom}} = \sqrt{\frac{\hbar G}{c^3}} \approx 1.61 \times 10^{-33}\ cm \equiv l_p}$$

g . Convert l_p to a mass + then an energy in eV

$$m_p = \sqrt{\frac{\hbar c}{G}} = \frac{l_p c^2}{G}$$

$$\rightarrow E_p = m_p c^2 = \sqrt{\frac{\hbar c^5}{G}} \approx 1.22 \times 10^{16}\ TeV.$$

The LHC is of the order TeV so this energy is above our current capabilities.